

Thermodynamical Metrics and Black Hole Phase Transitions

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ABSTRACT

An important phase transition in black hole thermodynamics is associated with the divergence of the specific heat with fixed charge and angular momenta, yet one can demonstrate that neither Ruppeiner's entropy metric nor Weinhold's energy metric reveals this phase transition. In this paper, we introduce a new thermodynamical metric based on the Hessian matrix of several free energy. We demonstrate, by studying various charged and rotating black holes, that the divergence of the specific heat corresponds to the curvature singularity of this new metric. We further investigate metrics on all thermodynamical potentials generated by Legendre transformations and study correspondences between curvature singularities and phase transition signals. We show in general that for a system with n -pairs of intensive/extensive variables, all thermodynamical potential metrics can be embedded into a flat (n, n) -dimensional space. We also generalize the Ruppeiner metrics and they are all conformal to the metrics constructed from the relevant thermodynamical potentials.

1 Introduction

Recently there has been considerable interest in applying the AdS/CFT correspondence [1] to understand certain aspects of condensed matter physics. In these studies, non-extremal black holes that are asymptotically anti-de Sitter (AdS) are the most natural backgrounds in the gravitational dual. This is due to the fact that a condensed matter system usually has a non-vanishing temperature, while gravitational configurations with temperature and appropriate asymptotic behaviors always lead to non-extremal black holes.

For asymptotically Minkowski spacetime, there can be one and only one Schwarzschild black hole at a given temperature. It has negative specific heat and evaporates via Hawking radiation. For an AdS Schwarzschild black hole, the story is different. The boundary of the AdS spacetime acts like a thermal box [2], and as such, black holes can only emerge above certain minimum temperatures. Above these minimum temperatures, there can be two types of black holes: the small ones are much like the usual Schwarzschild black hole of negative specific heat, whilst the big ones have positive specific heat and are locally stable. With the increasing of the temperature, the Helmholtz free energy of the large black holes can become less than the thermal background. This implies that above certain temperature pure thermal radiation in AdS becomes unstable and collapses to form black holes. This phase transition from the AdS thermal radiation to a black hole phase was discovered by Hawking and Page [3]. It turns out that the Hawking-Page phase transition can be interpreted, *via* the AdS/CFT correspondence, as a transition from a low-temperature confining phase to a high temperature de-confining phase in the boundary field theory [4].

It is natural to generalize the AdS Schwarzschild black hole to higher dimensions and to include charges and angular momenta. The thermodynamical properties of the Reissner-Nordström (RN) AdS black holes in arbitrary dimensions were studied in [5], and those of the Kerr-Newman-AdS black holes in [6]. It turns out that in those cases, locally stable black holes with positive specific heat exist for all temperature, including zero temperature which corresponds to the extremal limit. Increasing the temperature from zero, the (locally-stable) small black hole can undergo a first-order phase transition to become a large black hole, for charges or angular momenta less than certain critical values. This thermodynamical phase transition is different from the global Hawking-Page phase transition and it is characterized by the divergence of thermodynamical linear response function such as the specific heat. In this paper, we shall consider mainly this type of local thermodynamical phase transitions.

When the number of conserved quantities such as charges and angular momenta increases, the phase transitions become more and more complicated. In gauged supergravi-

ties, large classes of non-extremal charged rotating AdS black holes have been constructed [7, 8, 9, 10]. It would then be extremely useful to have a relative simple mechanism to identify phase transition points. One way is to study the thermodynamical geometry on the state space.

An important question now is how to construct metrics on this space. The simplest construction is based on the Hessian matrix of a certain quantity such as a thermodynamical potential, which is a function of either temperature or entropy together with other extensive variables. Consider a function f with variables x^i , $i = 1, \dots, n$, the elements of the corresponding Hessian matrix are $h_{i,j} = \partial^2 f / (\partial x^i \partial x^j)$. One can then construct a geometry with x^i as coordinates and $h_{i,j}$ as the metric components, *i.e.*

$$ds^2 = g_{ij} dx^i dx^j, \quad \text{with} \quad g_{ij} = h_{i,j}. \quad (1.1)$$

The first of such a metric in thermodynamics was introduced by Weinhold [11], making use of the first law of thermodynamics $dM = TdS + \sum \mu_i dN_i$, as the energy M is naturally a function of the entropy and extensive variables N_i . Ruppeiner later introduced a different metric, treating the entropy as the generating function depending on M and N^i [12]. At the first sight, Ruppeiner's entropy metric would appear to be most appropriate for black holes since the entropy, evaluated on the horizon, is rather different from the energy and N_i , which are all evaluated at the asymptotic infinity. However, it turns out that the Ruppeiner metric is related to the Weinhold's energy metric by an overall conformal factor $1/T$ [13]. There have been considerable amount of works in understanding these thermodynamical geometries of black holes, in particular to space-times which asymptote to AdS [14]-[26].

It should be mentioned right away that any geometry can only be non-trivial with at least two dimensions. However, the Hawking-Page phase transition of the AdS Schwarzschild black hole occurs in a one-dimensional system with only energy as its solution variable. This serves as a warning about the limitation of thermodynamical metrics. Nevertheless, studies of black holes with additional extensive variables have all painted a tantalizing picture relating curvature singularities of Ruppeiner or Weinhold metrics to phase transition points.

The signal of a phase transition typically appears when a capacity, such as the specific heat, charge capacitance or moment of inertia, changes the sign, which implies a change of stability. The sign change of these quantities can only happen by going through either zero or infinity. It was shown in [23] that whilst the Ruppeiner and Weinhold metrics indeed reveal the signals of black hole phase transitions associated with divergence of specific heat with fixed electric potential or angular velocity, they are insensitive to the Davies curve [27]

where the specific heat with fixed charge and/or angular momentum diverges. Various *ad hoc* modifications for the geometries were proposed [18, 22].

In this paper, we introduce new metrics based on the Hessian matrix of all thermodynamical potentials generated by Legendre transformations of the black hole energy or entropy. We calculate the corresponding Ricci scalars for various black holes and find that the collection of all curvature singularities are in one-to-one correspondence to the collection of threshold points of all capacities. In particular we find that the free-energy metric is always associated with the Davies curve.

The paper is organized as follows. In section 2, we present a general discussion of thermodynamical metrics. We show that all thermodynamical potential metrics including the Weinhold metric can be embedded into a higher-dimensional flat space. We also generalize the Ruppeiner metric and all these generalizations are conformal to the relevant potential metrics. In section 3, we consider Reissner-Nordström black holes in general dimensions. By examining all capacities including both specific heat C_Q and C_Φ and both charge capacitances \hat{C}_T and \hat{C}_S , we find that there are two threshold points where these quantities change sign by going through either infinity or zero. We calculate the Ricci scalar curvature for both Weinhold and Ruppeiner metrics and they share one same singularity that is precisely located at one of the threshold points. However, these two metrics are insensitive to the second threshold point. This prompts us to derive the Ricci scalar of the free-energy metric, and we find that its singularity corresponds to precisely the second threshold point. It is natural then to consider metrics generated by all thermodynamical potentials obtained by Legendre transformations. We find that they form conjugate pairs, in which the metrics are negative of each other. For RN AdS black holes, curvature singularities of Weinhold and free-energy metrics, obtained *via* Legendre transformations of the energy, give rise the to full set of threshold points.

In section 4, we consider the Kerr-Newman-AdS black hole in four dimensions. In this case, there are a total of twelve capacity-type quantities and eight thermodynamical potentials. We find that the full collection of threshold points of twelve capacities is exactly the same as that of curvature singularities of metrics associated with the eight potentials.

In section 5, we consider Kerr-AdS black holes. Owing to the complexity in the general case, we consider only four special examples. These are $D = 4$ and $D = 5$ solutions and general dimensional solutions of specific angular momenta. In section 6, we consider the Ricci-flat black ring solution with single angular momentum. Although details may be different in each case, essential features are of the same. We conclude our paper in section

7. In appendix A, we examine the van der Waals model and demonstrate that, as in the cases of black hole thermodynamics, additional metrics are needed and all threshold points can indeed be revealed. In appendices B and C, we present detailed results for discussions in sections 4 and 5, respectively.

2 General properties of thermodynamical geometry

In thermodynamical geometries, it is important to construct an appropriate metric for the equilibrium state space of a thermodynamical system. Let us start with the first law of thermodynamics

$$dM = TdS + \sum_{i=1}^n \mu_i dN_i. \quad (2.1)$$

Note that here we have adopted the black hole notations and used M to denote the “internal” energy of the system. The system has $(n+1)$ pairs of intensive/extensive variables (T, S) and (μ_i, N_i) . Here (μ_i, N_i) can be pairs of pressure/volume (P, V) , electric potential/charge (Φ, Q) , or angular velocity/momentum (Ω, J) , *etc.* The whole thermodynamical state space can be viewed as a $2(n+1)$ -dimensional embedding of the $(n+1)$ -dimensional space. The energy M is a function of $(n+1)$ extensive variables (S, N_i) . We find that the energy metric based on the Hessian matrix introduced by Weinhold can be simply written as

$$ds^2(M) = dTdS + \sum_{i=1}^n d\mu_i dN_i. \quad (2.2)$$

For black hole thermodynamics, this is a particularly convenient way to express the metric. This is because although the natural variables for M are S and N_i , the thermodynamical quantities in black holes are typically expressed in terms of parametric variables including the horizon radius. The explicit function of M in terms of S and N_i is not always available for a generic black hole. A metric like (2.2) removes the necessity of writing such an explicit function.

The first law of thermodynamics can also be expressed as

$$dS = \frac{1}{T}dM - \sum_{i=1}^n \frac{\mu_i}{T}dN_i. \quad (2.3)$$

It follows that the Ruppeiner entropy metric is given by

$$\begin{aligned} ds^2(S) &= d\left(\frac{1}{T}\right)dM - \sum_{i=1}^n d\left(\frac{\mu_i}{T}\right)dN_i \\ &= -\frac{1}{T}ds^2(M). \end{aligned} \quad (2.4)$$

as was demonstrated in [13].

We now generalize the Weinhold energy metric to those of other thermodynamical potentials yielded by Legendre transformations of internal energy M . For example, we may consider the Helmholtz free energy $F = M - TS$, as a function of T and N_i , which satisfies

$$dF = -SdT + \sum_{i=1}^n \mu_i dN_i. \quad (2.5)$$

The corresponding metric is given by

$$ds^2(F) = -dTdS + \sum_{i=1}^n d\mu_i dN_i. \quad (2.6)$$

Thus we can use parametric variables for thermodynamical quantities to write both the energy and free-energy metrics. The difference of these two metrics is simply a sign change of the term $dTdS$. This clearly simplifies the construction of metrics, by bypassing the need of obtaining explicit functions $M(S, N_i)$ and $F(T, N_i)$. By considering all possible thermodynamical potentials, we arrive at a total of 2^{n+1} metrics, given by

$$ds^2 = \pm dTdS + \sum_{i=1}^n \pm d\mu_i dN_i. \quad (2.7)$$

Here the plus and minus signs are independent. Since the overall “ $-$ ” factor is trivial, there are 2^n inequivalent metrics. We call thermodynamical potentials (U, \bar{U}) a conjugate pair if they satisfy

$$U + \bar{U} = 2M - TS - \sum_{i=1}^n \mu_i N_i. \quad (2.8)$$

Their associated metrics are negative of each other, *i.e.*

$$ds^2(U) = -ds^2(\bar{U}). \quad (2.9)$$

It is worth emphasizing that (2.7) implies that all thermodynamical potential metrics can be embedded in the $(n+1, n+1)$ -dimensional space, with light-cone coordinates (T, S) and (μ_i, N_i) . The thermodynamical geometries are $(n+1)$ -dimensional hyper-surfaces in this $(n+1, n+1)$ -dimensional flat space. The metrics (2.7) represent the totality of all possible flat embeddings. The sign choices in (2.7), except for the overall one, are non-trivial and cannot be absorbed by redefining the variables since it would imply different embedding functions.

Generalizations of the Ruppeiner metric are straightforward. They are all conformal to one of the potential metrics. For example, let us consider a thermodynamical potential U ,

satisfying

$$dU = \sum_{\alpha=0}^n \tilde{\mu}_\alpha d\tilde{N}_\alpha. \quad (2.10)$$

The U metric is given by $ds^2(U) = \sum_\alpha d\tilde{\mu}_\alpha d\tilde{N}_\alpha$. The Ruppeiner-like metric based on the Hessian matrix on the function \tilde{N}_β is then given by

$$ds^2(\tilde{N}_\beta) = -\frac{1}{\tilde{\mu}_\beta} ds^2(U). \quad (2.11)$$

Note that the natural variables for \tilde{N}_β are U and N_α 's with $\alpha \neq \beta$.

Having presented the general discussion of the thermodynamical metrics, we shall study some specific examples in the following sections.

3 Reissner-Nordström AdS black holes

Let us consider the Einstein-Maxwell theory with a negative cosmological constant in d dimensions. The Lagrangian is given by

$$\mathcal{L} = \sqrt{-g} \left(R - \frac{1}{4} F^2 + (d-1)(d-2)\lambda^2 \right), \quad (3.1)$$

where $F = dA$ is the field strength for the $U(1)$ vector potential A . Here, we decided to parameterized the cosmological constant by the inverse AdS radius, $\lambda \equiv \frac{1}{\ell}$. The theory admits an electrically-charged solution, often denominated in the literature Reissner-Nordström AdS black hole, given by

$$\begin{aligned} ds^2 &= -V dt^2 + \frac{dr^2}{V} + r^2 d\Omega_{d-2}^2, & A &= \frac{q\nu}{r^{d-3}} dt, \\ V &= 1 + \lambda^2 r^2 - \frac{m}{r^{d-3}} + \frac{q^2}{r^{2(d-3)}}, & \nu &= \sqrt{\frac{2(d-2)}{d-3}}. \end{aligned} \quad (3.2)$$

The mass m and the charge q are conserved quantities. The horizon of the black hole is located at $r = r_0$, which is the largest real root of V . For later convenience, we may express m in terms of r_0 ,

$$m = \frac{q^2}{r_0^{d-3}} + r_0^{d-3} (1 + \lambda^2 r_0^2). \quad (3.3)$$

It is straightforward to obtain thermodynamical quantities such as the Hawking temperature, entropy, electric potential, charge and mass. They are given by

$$\begin{aligned} T &= \frac{1}{4\pi} \left(\frac{d-3}{r_0} - \frac{(d-3)q^2}{r_0^{2d-5}} + (d-1)\lambda^2 r_0 \right), & S &= \frac{1}{4} r_0^{d-2} \omega_{d-2}, \\ \Phi &= \frac{q\nu}{r_0^{d-3}}, & Q &= \frac{(d-3)q\nu\omega_{d-2}}{16\pi}, \end{aligned}$$

$$M = \frac{(d-2)(q^2 r_0^6 + r_0^{2d}(1 + \lambda^2 r_0^2))\omega_{d-2}}{16\pi r_0^{d+3}}. \quad (3.4)$$

Here $\omega_{d-2} = 2\pi^{(d-1)/2}/(\frac{1}{2}(d-3))!$ is a pure numerical factor measuring the volume of the unit round $(d-2)$ -sphere. Note that these quantities satisfy the first law of thermodynamics,

$$dM = TdS + \Phi dQ. \quad (3.5)$$

An important quantity in thermodynamics is the specific heat defined by $C \equiv T(\partial S/\partial T)$. For Tangherlini-Schwarzschild black holes, corresponding to $q = 0$ and $\lambda = 0$ in our case, the specific heat is given by $C = -\frac{1}{4}(d-2)r_0^{d-2}\omega_{d-2}$, which is always negative and signaling a local thermodynamical instability. For AdS Schwarzschild black holes, it is given by

$$C = \frac{(d-2)\pi r_0^{d-1}\omega_{d-2}T}{(d-1)\lambda^2 r_0^2 - (d-3)}. \quad (3.6)$$

There is a minimal temperature $T_{\min} = \frac{1}{2\pi}\lambda\sqrt{(d-1)(d-3)}$, corresponding to $\lambda^2 r_0^2 = (d-3)/(d-1)$, above which black holes can tunnel to existence. This is also a temperature at which the above specific heat diverges. For a given temperature higher than T_{\min} , there could be two types of black hole solutions, the small ones with $\lambda^2 r_0^2 < (d-3)/(d-1)$ have negative specific heat, whilst the big ones with $\lambda^2 r_0^2 > (d-3)/(d-1)$ have positive specific heat and are locally stable. The Helmholtz free energy is given by

$$F = \frac{1}{16\pi}r_0^{d-3}(1 - \lambda^2 r_0^2). \quad (3.7)$$

When the temperature is above $T = \frac{1}{2\pi}(d-2)\lambda$, corresponding to $\lambda r_0 > 1$, the large locally-stable black holes start to have negative free energy and hence are more probable than the pure AdS radiation [3].

For general RN AdS black holes, their phase transitions were studied in detail in [5]. Owing to the existence of charges, there can be extremal and near-extremal black holes, which are locally stable. Locally stable RN black holes can exist at all temperature. When the temperature increases, small black holes can have a phase transition to become a large black hole with discontinuous entropy. This is rather different from the global Hawking-Page phase transition and is characterized by the divergence of the specific heat. To be specific, one may define the specific heat for either constant charge Q or constant electric potential Φ , given by

$$\begin{aligned} C_Q &\equiv T \frac{\partial S}{\partial T} \Big|_Q = \frac{\partial M}{\partial T} \Big|_Q = \frac{(d-2)\pi r_0^{d-1}\omega_{d-2}T}{\zeta_1}, \\ C_\Phi &\equiv T \frac{\partial S}{\partial T} \Big|_\Phi = -\frac{(d-2)\pi r_0^{3d-1}\omega_{d-2}T}{\zeta_2}. \end{aligned} \quad (3.8)$$

where

$$\begin{aligned}\zeta_1 &= (d-1)\lambda^2 r_0^2 - (d-3)(1 - (2d-5)r_0^{2(3-d)}q^2), \\ \zeta_2 &= r_0^{2d}(d-3 - (d-1)\lambda^2 r_0^2) - (d-3)q^2 r_0^6.\end{aligned}\tag{3.9}$$

One may also introduce analogously charge capacitances at fixed temperature or entropy, given by

$$\begin{aligned}\tilde{C}_T &\equiv \left. \frac{\partial Q}{\partial \Phi} \right|_T = \frac{(d-3)r_0^{3(d-1)}\omega_{d-2}\zeta_1}{16\pi\zeta_2}, \\ \tilde{C}_S &\equiv \left. \frac{\partial Q}{\partial \Phi} \right|_S = \frac{1}{16\pi}(d-3)r_0^{d-3}\omega_{d-2}.\end{aligned}\tag{3.10}$$

We see that \tilde{C}_S is positive definite while the other three C 's may change signs. There are two threshold points,

$$\zeta_1 = 0, \quad \text{and} \quad \zeta_2 = 0.\tag{3.11}$$

For non-vanishing q , ζ_1 and ζ_2 cannot vanish simultaneously. For zero cosmological constant, the condition for $\zeta_2 = 0$ coincides with $T = 0$. The solution becomes extremal and the mass/charge relation is saturated, *i.e.* $M = M_{\min} \equiv \nu Q$. The condition for $\zeta_1 = 0$ implies that

$$M = \sqrt{\frac{2(d-2)^3}{(d-3)(2d-5)}} Q > M_{\min}.\tag{3.12}$$

These threshold points are of two types. For one type, the corresponding specific heat and/or charge capacitance changes sign by going through infinity. This is the case for C_Φ and \tilde{C}_T at $\zeta_2 = 0$, and also C_Q at $\zeta_1 = 0$. For the other type, a capacity changes sign by going through zero, as \tilde{C}_T at $\zeta_1 = 0$. For all black hole examples we have examined in this paper, the vanishing of a certain capacity is always associated with the divergence of another. Thus we shall not be very strict on distinguishing the divergent or vanishing points. As we shall see in section 6, the situation for the five-dimensional black ring is somewhat different.

Such threshold points signal the change of stability and hence are typically associated with certain phase transitions. The phase transition associated with the divergence of C_Q was well studied in [5]. A new type of phase transition for the RN black hole associated with the divergence of C_Φ was recently suggested in [25, 26]. It is natural to expect that these important phase-transition points can be seen in a thermodynamical metric, as curvature singularities. As discussed in section 2, the Weinhold and Ruppeiner metrics can be written as

$$ds^2(M) = dT dS + d\Phi dQ, \quad ds^2(S) = -\frac{1}{T} ds^2(M).\tag{3.13}$$

Their Ricci scalars can be readily calculated, given by

$$\begin{aligned} R^{(M)} &= \frac{16\pi(d-3)^2 r_0^{3(d+1)}}{(d-2)\omega_{d-2}\zeta_2^2}, \\ R^{(S)} &= \frac{(d-1)\lambda^2 R^{(M)}}{16\pi^2(d-3)^2 r_0^{4d} T} \left(3(d-3)q^2 r_0^6 - r_0^{2d}(3(d-3) - (d-1)\lambda^2 r_0^2) \right) \times \\ &\quad \left((d-3)(d-2)q^2 r_0^6 + r_0^{2d}((d-3)(d-4) + (d-1)(d-2)\lambda^2 r_0^2) \right). \end{aligned} \quad (3.14)$$

We see that the curvature singularity of these metrics are related to the phase transition associated with the vanishing of ζ_2 , but not ζ_1 . In Ruppeiner geometry, there is an additional curvature singularity at $T = 0$, corresponding to the extremal limit of the black holes. It is worth remarking that when the cosmological constant parameter $\lambda = 0$, the Ruppeiner curvature vanishes and hence reveals no information at all.

The inability of the Weinhold and Ruppeiner metrics to probe the phase transition associated with the vanishing of ζ_1 prompts us to look for other types of thermodynamical geometries. We construct a new metric based on the Hessian matrix of the free energy, defined by

$$F(T, Q) = M - TS, \quad dF = -SdT + \Phi dQ. \quad (3.15)$$

For RN AdS black holes, we have

$$F = \frac{(2d-5)q^2 r_0^6 + r_0^{2d}(1 - \lambda^2 r_0^2)}{16\pi r_0^{d+3}}. \quad (3.16)$$

As discussed in section 2, it is not always convenient nor necessary to use natural variables (T, Q) to construct the free-energy metric. We shall use the (r_0, q) variables and the flat-space embedding metric,

$$ds^2(F) = -dT dS + d\Phi dQ. \quad (3.17)$$

Substituting the thermodynamical quantities in (3.4), we obtain the free-energy metric with (r_0, q) coordinates. The Ricci scalar for the free-energy metric is then given by

$$R^{(F)} = \frac{16(d-3)^2 \pi r_0^{3(d+1)} (d-2 - (d-1)\lambda^2 r_0^2)}{(d-2)\omega_{d-2}\zeta_1^2}. \quad (3.18)$$

The curvature singularity corresponds precisely to the phase transition at $\zeta_1 = 0$, associated with the divergence of the specific heat C_Q .

We have also investigated other Legendre transformations, namely $\bar{F}(S, \Phi) = M - \Phi Q$ and $\bar{M}(T, \Phi) = M - TS - \Phi Q$. As discussed in section 2, they are conjugates to F and M respectively. It follows then

$$R^{(\bar{F})} = -R^{(F)}, \quad R^{(\bar{M})} = -R^{(M)}. \quad (3.19)$$

There are no further curvature singularities by considering these additional metrics. This is encouraging since we do not expect any further phase transitions in the RN AdS black holes. The totality of curvature singularities of all metrics is exactly the same as the totality of capacity threshold points. In [18], a generalized Ruppeiner metric was considered with variables (M, Q) replaced by $(M - \Phi Q, \Phi)$. It follows from the discussion in section 2, that metric is conformal to our $ds^2(F) = -ds^2(\bar{F})$ with a conformal factor $1/T$. It has the same singularity as the one associated with the divergence of C_Q .

It should be remarked that although thermodynamical geometries reveal precisely the signals of phase transitions of the first order associated with the divergence of capacities, it lacks the power to reveal conveniently subtle points such as the emergence of the second order phase transitions. It follows from (3.4) that we can treat the parameters (r_0, q) as describing independently the entropy and charge respectively. It is clear from the temperature expression in (3.4) that in general there are two threshold points associated with the divergence of the specific heat at the fixed charge Q . When the two minima of $T(S)$ merge, the first-order phase transitions degenerate into a second order one. This occurs when $q = q'$ and $r_0 = r'_0$, given by

$$q'^2 = \frac{r_0'^{2(d-3)}}{(d-2)(2d-5)}, \quad r_0'^2 = \frac{(d-3)^2}{(d-1)(d-2)\lambda^2}. \quad (3.20)$$

When $q > q'$, the specific heat is always positive and non-infinity, and hence there is no phase transition. When $q = q'$, the specific heat is always positive and there is a second-order phase transition occurs whenever

$$T = \frac{(d-3)\sqrt{(d-2)(d-1)}\lambda}{(2d-5)\pi}. \quad (3.21)$$

For $0 < q < q'$, there is a phase transition of the first order.

4 Kerr-Newman-AdS black holes

An analytical solution for the rotating charged black holes of Einstein-Maxwell theories with or without cosmological constant is only known for $d = 4$, and it is called the Kerr-Newman-(AdS) black hole [28]. Charged rotating black holes in five-dimensional minimum gauged supergravity are also known [9]. However, this theory contains an additional Chern-Simons term.

The thermodynamical properties for the Kerr-Newman-AdS black hole were analyzed

in [6]. The thermodynamical quantities are given by

$$\begin{aligned} T &= \frac{r_0^2(1 + (3r_0^2 + a^2)\lambda^2) - q^2 - a^2}{4\pi r_0(r_0^2 + a^2)}, & S &= \frac{\pi(r_0^2 + a^2)}{\Xi}, \\ \Omega &= \frac{a(1 + \lambda^2 r_0^2)}{r_0^2 + a^2}, & J &= \frac{am}{\Xi^2}, & \Phi &= \frac{qr_0}{r_0^2 + a^2}, & Q &= \frac{q}{\Xi}, & M &= \frac{m}{\Xi^2}, \end{aligned} \quad (4.1)$$

where the parameters m and Ξ are given by

$$m = \frac{(r_0^2 + a^2)(1 + \lambda^2 r_0^2) + q^2}{2r_0}, \quad \Xi = 1 - \lambda^2 a^2. \quad (4.2)$$

The specific heat at constant angular momentum and charge is given by [6]

$$C_{J,Q} = \frac{4\pi MTS}{1 - 4\pi T(2M + TS) + 2\lambda^2(Q^2 + 3\pi^{-1}S) + 6\pi^{-2}\lambda^4 S^2}. \quad (4.3)$$

The Ruppeiner geometry of the Kerr-Newman black hole was studied in [23]. The curvature is not sensitive to the Davies curve where the specific heat $C_{J,Q}$ diverges. It follows from the previous discussion that this phase transition is expected to be related to the curvature singularity of the free-energy metric. Indeed, it is straightforward to verify that the denominator for the Ricci scalar $R^{(F)}$ is the same as that of $C_{J,Q}$ up to trivial non-vanishing factors.

In the following, we shall use $D(X)$ to denote the denominator of a quantity X . The denominator of the Ricci scalar of the Weinhold metric is given by

$$D(R^{(M)}) = 4m^2 r_0^2 \zeta^2, \quad (4.4)$$

where

$$\zeta = (r_0^2 + a^2)(1 + \lambda^2 r_0^2)(r_0^2 + a^2 + \lambda^2 r_0^2(a^2 - 3r_0^2)) + q^2(3a^2 - r_0^2 - \lambda^2 r_0^2(r_0^2 + a^2)). \quad (4.5)$$

As one would have expected, this factor appears precisely in the denominators of both charge capacitance $\tilde{C}_{T,\Omega}$ and moment of inertia $\hat{C}_{T,\Phi}$, defined by¹

$$\tilde{C}_{T,\Omega} = \frac{\partial Q}{\partial \Phi} \Big|_{T,\Omega}, \quad \hat{C}_{T,\Phi} = \frac{\partial J}{\partial \Omega} \Big|_{T,\Phi}. \quad (4.6)$$

The denominator of the Ricci scalar in the Ruppeiner geometry is

$$D(R^{(S)}) = 4\pi^2 r_0(r_0^2 + a^2)T D(R^{(M)}). \quad (4.7)$$

¹We use C, \tilde{C} and \hat{C} to denote specific heat, charge capacitance and moment of inertia, respectively. The subscripts denote quantities that are held fixed.

Note that there are a total of eight thermodynamical potentials related by Legendre transformations. To systematically study curvature singularities and capacity threshold points, we label them as follows

$$\begin{aligned}
M, \quad \bar{M} &= M - TS - \Phi Q - \Omega J, \\
F &= M - TS, \quad \bar{F} = M - \Phi Q - \Omega J, \\
H &= M - \Phi Q, \quad \bar{H} = M - TS - \Omega J, \\
L &= M - \Omega J, \quad \bar{L} = M - TS - \Phi Q,
\end{aligned} \tag{4.8}$$

Following discussions in section 2, the barred and unbarred quantities are conjugate pairs. There are a total of four independent Ricci scalar quantities, and each one gives rise to one singularity, resulting in total of four different singularities. There are twelve capacities: four specific heat, four charge capacitances, and four moments of inertia. There are eight capacities that involve a total of four threshold points, corresponding precisely the four curvature singularities. The relationship between curvature singularities and capacity threshold points can be summarized as follows

$$\begin{aligned}
R^{(\bar{M})} = -R^{(M)} &\longrightarrow \frac{\partial Q}{\partial \Phi} \Big|_{T, \Omega} \sim \frac{\partial J}{\partial \Omega} \Big|_{T, \Phi} \sim T \frac{\partial S}{\partial T} \Big|_{\Phi, \Omega}, \\
R^{(\bar{F})} = -R^{(F)} &\longrightarrow \frac{\partial M}{\partial T} \Big|_{Q, J}, \\
R^{(\bar{H})} = -R^{(H)} &\longrightarrow \frac{\partial J}{\partial \Omega} \Big|_{T, Q} \sim T \frac{\partial S}{\partial T} \Big|_{Q, \Omega}, \\
R^{(\bar{L})} = -R^{(L)} &\longrightarrow \frac{\partial Q}{\partial \Phi} \Big|_{T, J} \sim T \frac{\partial S}{\partial T} \Big|_{\Phi, J}.
\end{aligned} \tag{4.9}$$

Here “ \sim ” links terms which have the same pole. The remaining quantities $\partial Q/\partial \Phi \Big|_{S, J}$, $\partial Q/\partial \Phi \Big|_{S, \Omega}$, $\partial J/\partial \Omega \Big|_{S, \Phi}$ and $\partial J/\partial \Omega \Big|_{S, Q}$ have no threshold points. We see that all possible phase transitions correspond to curvature singularities of certain thermodynamical metrics. See appendix B for explicit results.

5 Kerr-AdS black holes

General Kerr-AdS black holes were constructed in [29]. Their thermodynamical quantities were obtained in [30], which we adopt for our calculation. Their general thermodynamical geometries are complicated. We shall consider only some special and simple cases.

Four dimensions:

Although this is a special case of the Kerr-Newman-AdS black hole discussed in the previous section, it is instructive to list it here. The thermodynamical metrics are two-dimensional, and we find the following scalar curvatures for the $F(T, J)$, $M(S, J)$ and $S(M, J)$ metrics:

$$R^{(F)} = \frac{P_1(r_0, a, \lambda)}{r_0 \zeta_1^2}, \quad R^{(M)} = \frac{\lambda^2 P_2(r_0, a, \lambda)}{(r_0^2 + a^2) \zeta_2^2}, \quad R^{(S)} = \frac{P_3(r_0, a, \lambda)}{4\pi^2 (r_0^2 + a^2) T \zeta_2^2}, \quad (5.1)$$

where P_i are non-singular polynomial functions of r_0, a, λ , and

$$\begin{aligned} \zeta_1 &= 3a^4 + 6a^2 r_0^2 - r_0^4 + (a^6 + 13a^4 r_0^2 + 23a^2 r_0^4 + 3r_0^6) \lambda^2 + a^2 r_0^2 (a^2 + 3r_0^2)^2 \lambda^4, \\ \zeta_2 &= -r_0^2 - a^2 + \lambda^2 r_0^2 (3r_0^2 - a^2). \end{aligned} \quad (5.2)$$

All specific heats and moments of inertia are given by

$$\begin{aligned} C_J &\equiv T \frac{\partial S}{\partial T} \Big|_J = \frac{8\pi r_0 (r_0^2 + a^2)^3 T}{(1 - a^2 \lambda^2) \zeta_1}, & C_\Omega &\equiv T \frac{\partial S}{\partial T} \Big|_\Omega = \frac{8\pi^2 r_0^3 (r_0^2 + a^2) T}{(1 - a^2 \lambda^2) \zeta_2}, \\ \hat{C}_T &\equiv \frac{\partial J}{\partial \Omega} \Big|_T = \frac{(r_0^2 + a^2) \zeta_1}{2r_0 (1 - a^2 \lambda^2)^3 \zeta_2}, & \hat{C}_S &\equiv \frac{\partial J}{\partial \Omega} \Big|_S = \frac{(r_0^2 + a^2)^3}{2r_0^3 (1 - a^2 \lambda^2)^3}, \end{aligned} \quad (5.3)$$

The relationship between curvature singularities and capacity thresholds is clear.

Note that the λ^2 factor in $R^{(M)}$ implies that the Ricci scalar for the Weinhold metric vanishes for zero cosmological constant. It appears that the Weinhold metric fails to predict the phase transition associated with the divergence of the moment of inertia. However, as we can see from (5.2) that ζ_2 has no zero when $\lambda = 0$, and hence \hat{C}_T will not diverge.

Five dimensions:

In this case, there are three conserved quantities, mass and two angular momenta, parameterized by r_0 and a and b . The scalar curvature for the Weinhold metric in this case diverges when

$$r_0^4 (1 + \lambda^2 (a^2 + b^2 - 2r_0^2)) - a^2 b^2 = 0. \quad (5.4)$$

As in the previous case, this divergence coincides with the divergence of the moments of inertia, defined by

$$\hat{C}_{T, \Omega_b} \equiv \frac{\partial J_a}{\partial \Omega_a} \Big|_{T, \Omega_b}, \quad \hat{C}_{T, \Omega_a} \equiv \frac{\partial J_b}{\partial \Omega_b} \Big|_{T, \Omega_a}. \quad (5.5)$$

They turn out to have the same divergent pole. For the Ruppeiner geometry, we find the same relationship as (4.7).

A new feature arises in the free-energy metric. There are two singularities in the Ricci scalar $R^{(F)}$. One is associated with the divergence of the specific heat C_{J_a, J_b} . The other is located at

$$a^2 b^2 + (a^2 + b^2) r^2 - 3r_0^4 = 0, \quad (5.6)$$

which does not explicitly depend on the cosmological constant. It turns out that this point corresponds to the divergence of the moments of inertia defined by

$$\hat{C}_{S,\Omega_b} \equiv \frac{\partial J_a}{\partial \Omega_a} \Big|_{S,\Omega_b}, \quad \hat{C}_{S,\Omega_a} \equiv \frac{\partial J_b}{\partial \Omega_b} \Big|_{S,\Omega_a}. \quad (5.7)$$

Both moments of inertia have the same divergent pole.

To summarize, there are a total of eight thermodynamical potentials related to each other by the Legendre transformations, given by

$$\begin{aligned} M, \quad \bar{M} &= M - TS - \Omega_a J_a - \Omega_b J_b, \\ F &= M - TS, \quad \bar{F} = M - \Omega_a J_a - \Omega_b J_b, \\ L_a &= M - \Omega_a J_a, \quad \bar{L}_a = M - TS - \Omega_b J_b, \\ L_b &= M - \Omega_b J_b, \quad \bar{L}_b = M - TS - \Omega_a J_a. \end{aligned} \quad (5.8)$$

The following are the relations between curvature singularities and capacity thresholds:

$$\begin{aligned} R^{(M)} = -R^{(\bar{M})} &\longrightarrow T \frac{\partial S}{\partial T} \Big|_{\Omega_a, \Omega_b} \sim \frac{\partial J_a}{\partial \Omega_a} \Big|_{T, \Omega_b} \sim \frac{\partial J_b}{\partial \Omega_b} \Big|_{T, \Omega_a}, \\ R^{(F)} = -R^{(\bar{F})} &\longrightarrow T \frac{\partial S}{\partial T} \Big|_{J_a, J_b}, \quad \frac{\partial J_a}{\partial \Omega_a} \Big|_{S, \Omega_b} \sim \frac{\partial J_b}{\partial \Omega_b} \Big|_{S, \Omega_a}, \\ R^{(L_a)} = -R^{(-\bar{L}_a)} &\longrightarrow \frac{\partial J_a}{\partial \Omega_a} \Big|_{S, J_b}, \quad T \frac{\partial S}{\partial T} \Big|_{J_a, \Omega_b} \sim \frac{\partial J_b}{\partial \Omega_b} \Big|_{T, J_a}, \\ R^{(L_b)} = -R^{(-\bar{L}_b)} &\longrightarrow \frac{\partial J_b}{\partial \Omega_b} \Big|_{S, J_a}, \quad T \frac{\partial S}{\partial T} \Big|_{J_b, \Omega_a} \sim \frac{\partial J_a}{\partial \Omega_a} \Big|_{T, J_b}. \end{aligned} \quad (5.9)$$

The Weinhold metric and its conjugate pair have one singularity whilst the remaining 6 metrics all have two singularities. It is important to emphasize that all possible phase transition signals, associated with the divergence of any of the twelve capacities, are captured by curvature singularities. Conversely, there is not a single curvature singularity that is not related to a phase transition signal. Detailed results can be found in appendix C.

Kerr-AdS with a single angular momentum:

In this case, we take $a_1 = a$ and $a_i = 0$ for $i \geq 2$. As in the general $D = 5$ case, there are two curvature singularities in the metric associated with the free-energy. One corresponds to the divergence of the specific heat C_J , whilst the other occurs when we have

$$\begin{aligned} d = 2N + 1 : \quad r_0^2 &= \frac{(2N - 3)a^2}{2N - 1}, \\ d = 2N + 2 : \quad r_0^2 &= \frac{(N - 1)a^2}{N}. \end{aligned} \quad (5.10)$$

This point corresponds to the divergence of the moment of inertia

$$\hat{C}_S = \left. \frac{\partial J_a}{\partial \Omega_a} \right|_S. \quad (5.11)$$

The Weinhold and Ruppeiner metrics give no new results. We have $R^{(M)} \sim C_\Omega \sim \hat{C}_T$ and $D(R^{(R)}) = TD(R^{(M)})$.

Kerr-AdS with all equal angular momenta:

In this case, we take $a_i = a$ for all i . We find the same correspondence between the free-energy metric and the specific heat C_J . There appears to be a correspondence between the curvature singularity and the divergence of the capacitance C_S , since both the Ricci scalar and the capacitance have the same factor in the denominator, namely $(2N-1)r_0^2 + a^2$. However, this factor can never be zero. The result for the Weinhold and Ruppeiner geometry is the same as that of previous examples.

6 Black Ring

The black ring solution in five dimensions with single angular momentum was first obtained in [31]. We shall adopt the notation of [32], and the thermodynamical quantities are given by

$$\begin{aligned} T &= \frac{1}{2\pi} \eta_2 (\eta_1 + \eta_2), & S &= \frac{\pi^2}{2\eta_2 (\eta_1^2 - \eta_2^2) (\eta_1 + \eta_2)^3}, \\ \Omega &= (\eta_1^2 - \eta_2^2) \sqrt{\frac{\eta_2}{\eta_1}}, & J &= \frac{\pi \eta_1^{\frac{3}{2}}}{4\eta_2^{\frac{3}{2}} (\eta_1^2 - \eta_2^2)^2 (\eta_1 + \eta_2)^2}, \end{aligned} \quad (6.1)$$

where $\eta_1 > \eta_2 > 0$ and $\eta_1 \eta_2 < 1$. The Ruppeiner geometry of this system was studied in [33].

In this case, we find that the Weinhold metric is flat. This should imply that there is no phase transition associated with the moment of inertia at constant temperature. It is indeed the case, as we shall see next. The Ricci scalars for the free-energy and Ruppeiner metrics are given by

$$\begin{aligned} R^{(F)} &= -\frac{192\eta_1\eta_2(\eta_1 - \eta_2)^2(\eta_1 + \eta_2)^6(\eta_1^2 - 2\eta_1\eta_2 + 9\eta_2^2)}{\pi(\eta_1^2 - 6\eta_1\eta_2 - 3\eta_2^2)^2(3\eta_1^2 + 2\eta_1\eta_2 + 3\eta_2^2)^2}, \\ R^{(S)} &= -\frac{2\eta_2(\eta_1 - \eta_2)^2(\eta_1 + \eta_2)^3(\eta_1^2 - 6\eta_1\eta_2 - 3\eta_2^2)}{\pi^2(\eta_1 - 3\eta_2)^2}. \end{aligned} \quad (6.2)$$

Each gives a singularity, corresponding to $\eta_1^2 - 6\eta_1\eta_2 - 3\eta_2^2 = 0$ or $\eta_1 = 3\eta_2$. These points should be related to certain phase transitions. For the Emparan-Reall black ring, one

can readily calculate both the specific heat at constant angular momentum J and angular velocity Ω ,

$$\begin{aligned} C_J &= \frac{3\pi^2(\eta_1 - 3\eta_2)}{2(\eta_1 - \eta_2)\eta_2(\eta_1 + \eta_2)^3(3\eta_1^2 + 2\eta_1\eta_2 + 3\eta_2^2)}, \\ C_\Omega &= \frac{\pi^2(\eta_1^2 - 6\eta_1\eta_2 - 3\eta_2^2)}{2\eta_2(\eta_1 - \eta_2)(\eta_1 + \eta_2)^6}, \end{aligned} \quad (6.3)$$

and moments of inertia at constant T or S :

$$\begin{aligned} \hat{C}_T &= -\frac{\pi\eta_1^2(3\eta_1^2 + 2\eta_1\eta_2 + 3\eta_2^2)}{4\eta_2^2(\eta_1 - \eta_2)^3(\eta_1 + \eta_2)^7}, \\ \hat{C}_S &= -\frac{3\pi\eta_1^2(\eta_1 - 3\eta_2)}{4(\eta_1 - \eta_2)^3\eta_2^2(\eta_1 + \eta_2)^4(\eta_1^2 - 6\eta_1\eta_2 - 3\eta_2^2)}. \end{aligned} \quad (6.4)$$

We note that \hat{C}_T has neither pole nor zero, consistent with the flat Weinhold geometry. The singularity of $R^{(F)}$ corresponds to the divergence of \hat{C}_S , not C_J . This is very different from previous black hole thermodynamics. Although $R^{(F)}$ and C_J share a common denominator, it does not vanish. Another new feature is that curvature singularity of the Ruppeiner metric corresponds to the zero of C_J and \hat{C}_S , instead of a typical divergent point.

7 Conclusions

A geometrical understanding of the state space of a thermodynamical system has been proposed for thirty years. Its application in black hole thermodynamics has generated interesting results. An important and difficult question is how to construct an appropriate metric over the space. People have been focusing on mainly two metrics, the energy metric proposed by Weinhold and the entropy metric proposed by Ruppeiner.

By investigating various black holes, we find that although the energy and entropy metrics can reveal certain phase transitions at their curvature singularities, they miss some important ones, including the phase transition along the Davies curve where the specific heat with constant charge and angular momentum diverges. We generalize this procedure and consider other thermodynamical potentials via Legendre transformations as the seed function for the Hessian matrix. We find that the curvature singularity of the free-energy metric is always located along the Davies curve. We obtain the full set of curvature singularities for all the metrics generated by thermodynamical potentials. We find that they form conjugate pairs: the metrics are negative of each other.

A black hole in a supergravity theory can typically involve many conserved quantities, including the mass, charges and angular momenta. There are three types of capacities:

the specific heat, charge capacitances and momenta of inertia. There may exist a few threshold points where these capacities change sign, signaling a change of stability and phase transitions. For all black holes and a black ring we have examined, we find that the full collection of these thresholds is exactly the same as the full collection of the curvature singularities. We expect that this is true for all thermodynamical systems. On the other hand, we did not find a thermodynamical metric which could signal all of the thermodynamic phase transitions. This should not be surprising since different thermodynamical phase transitions occur in very different ensembles. It is reasonable to expect that only the metric of an appropriate thermodynamical potential is relevant to a particular phase transition.

A thermodynamical system with n pairs of intensive/extensive variables (μ_i, N_i) have an n -dimensional state space that can be viewed as an embedding space of $2n$ -dimensional flat space with (μ_i, N_i) as coordinates. There exist a total of 2^{n-1} inequivalent thermodynamical potential metrics based on the Hessian matrix, which are in turn possible to embed in flat (n, n) -dimensional space. This provides a rather simple geometric picture of thermodynamical state space. Furthermore it also provides a very simple mechanism to compute the metrics and curvature invariants. The questions of the physical significance of these metrics and what is the principle underling their relationship with phase transitions remain open.

Acknowledgement

We are grateful to Jim Ferguson and Pang Yi for useful discussions, and to KITPC for hospitality. We are also grateful to the referee for proofreading our manuscript. H. Liu and M. Luo are supported in part by the National Science Foundation of China (10425525, 10875103), National Basic Research Program of China (2010CB833000), and Zhejiang University Group Funding (2009QNA3015).

A The Van der Waals Model

In this appendix, we consider the thermodynamical geometry for the Van der Waals model. Its similarity with the black hole thermodynamics was noted in [24]. We shall demonstrate, as in the case of black holes, additional thermodynamical metrics are necessary to probe all the phase transitions. Adopting the notation in [24], the free-energy for the Van der Waals

model is given by

$$F = N(-c_v T \log T - \zeta T + \epsilon) - NT \log(e(V - Nb)N^{-1}) - N^2 a V^{-1}, \quad (\text{A.1})$$

where F is a function of T and V , satisfying

$$dF = -SdT + PdV. \quad (\text{A.2})$$

For simplicity, we set $\zeta = 1$ and $\epsilon = 1$. The entropy and pressure are given by

$$S = N(c_v(1 + \log T) + \log(e(V - Nb)N^{-1})), \quad P = N\left(\frac{T}{Nb - V} + \frac{Na}{V^2}\right). \quad (\text{A.3})$$

The (P, V) pair is very much like the (Φ, Q) or (Ω, J) pairs in black hole physics. There are a total of four types of capacities, given by

$$\begin{aligned} C_V &= T \frac{\partial S}{\partial T} \Big|_V = c_v, \\ C_P &= T \frac{\partial S}{\partial T} \Big|_P = \frac{N((1 + c_v)TV^3 - 2ac_v N(Nb - V)^2)}{TV^3 - 2Na(Nb - V)^2}, \\ \frac{\partial V}{\partial P} \Big|_T &= \frac{(Nb - V)^2 V^3}{Na(TV^3 - 2Na(Nb - V)^2)}, \\ \frac{\partial V}{\partial P} \Big|_S &= \frac{c_v V^3 (Nb - V)^2}{Na((1 + c_v)TV^3 - 2ac_v N(Nb - V)^2)}. \end{aligned} \quad (\text{A.4})$$

Similar to RN AdS black holes, the system has two threshold points. One difference is that the C_V here is a constant, whilst the corresponding C_Q of an RN black hole has a divergent point, associated with the Davies curve. Note that since the factor $(Nb - V)^2$ is non-negative, the vanishing of this factor does not correspond to a threshold point. Also we find that both Ruppeiner and Weinhold curvature reveals only one threshold, associated with C_P , as in the case of the black hole cases discussed in the paper. To be specific, we have

$$\begin{aligned} R^{(M)} &= \frac{aV^3(Nb - V)^2}{c_v(TV^3 - 2Na(Nb - V)^2)^2}, \\ R^{(S)} &= -\frac{2a(Nb - V)^2(TV^3 - Na(Nb - V)^2)}{c_v(TV^3 - 2Na(Nb - V)^2)^3}. \end{aligned} \quad (\text{A.5})$$

The other divergent point can be revealed by the free-energy metric, *i.e.*

$$R^{(F)} = \frac{ac_v(3Nb - 2V)V^2(Nb - V)^2}{\left((1 + c_v)TV^3 - 2ac_v N(Nb - V)^2\right)^2}. \quad (\text{A.6})$$

B Capacities of Kerr-Newman AdS Black holes

In this appendix, we present some detailed results that were discussed in section 4. The thermodynamical curvatures are in general too complex to present. Here we only give the poles:

$$\begin{aligned} R^{(M)} &= -R^{(\bar{M})} \sim \frac{1}{\zeta_1^2}, & R^{(F)} &= -R^{(\bar{F})} \sim \frac{1}{\zeta_2^2}, \\ R^{(L)} &= -R^{(\bar{L})} \sim \frac{1}{\zeta_3^2}, & R^{(H)} &= -R^{(\bar{H})} \sim \frac{1}{\zeta_4^2}, \end{aligned} \quad (\text{B.1})$$

where

$$\begin{aligned} \zeta_1 &= -(r_0^2 + a^2)(1 + \lambda^2 r_0^2)(r_0^2 + a^2 + \lambda^2 r_0^2(a^2 - 3r_0^2)) + q^2(r_0^2 - 3a^2 + \lambda^2 r_0^2(r_0^2 + a^2)), \\ \zeta_2 &= 3a^4 + q^4(3 + a^2\lambda^2) + 16a^4\lambda^2 r_0^2 + 19a^4\lambda^4 r_0^4 + 6a^4\lambda^6 r_0^6 + a^6(\lambda + \lambda^3 r_0^2)^2 \\ &\quad + r_0^4(-1 + 2\lambda^2 r_0^2 + 3\lambda^4 r_0^4) + a^2 r_0^2(6 + 29\lambda^2 r_0^2 + 32\lambda^4 r_0^4 + 9\lambda^6 r_0^6) \\ &\quad + q^2(a^4(2\lambda^2 - 2\lambda^4 r_0^2) + 2(r_0^2 + 3\lambda^2 r_0^4) + a^2(6 + 8\lambda^2 r_0^2 - 6\lambda^4 r_0^4)), \\ \zeta_3 &= a^8\lambda^2(1 + \lambda^2 r_0^2)^2 + a^6(3 + 4q^2\lambda^2 + 17\lambda^2 r_0^2 + 21\lambda^4 r_0^4 + 7\lambda^6 r_0^6) \\ &\quad + r_0^2(q^4 - r_0^4 + 4q^2\lambda^2 r_0^4 + 2\lambda^2 r_0^6 + 3\lambda^4 r_0^8) \\ &\quad + 3a^4(q^4\lambda^2 + 4q^2(1 + 3\lambda^2 r_0^2 + \lambda^4 r_0^4) + r_0^2(3 + 15\lambda^2 r_0^2 + 17\lambda^4 r_0^4 + 5\lambda^6 r_0^6)) \\ &\quad + a^2(3q^4(3 + \lambda^2 r_0^2) + 12q^2(r_0^2 + 3\lambda^2 r_0^4 + \lambda^4 r_0^6) \\ &\quad + r_0^4(5 + 31\lambda^2 r_0^2 + 35\lambda^4 r_0^4 + 9\lambda^6 r_0^6)), \\ \zeta_4 &= -a^4(1 + \lambda^2 r_0^2)^2 + r_0^2(1 + \lambda^2 r_0^2)(3q^2 - r_0^2 + 3\lambda^2 r_0^4) \\ &\quad - a^2(2r_0^2 - 2\lambda^4 r_0^6 + q^2(1 - 3\lambda^2 r_0^2)). \end{aligned} \quad (\text{B.2})$$

There are a total of twelve capacities: four specific heat, four charge capacitances and four moments of inertia. It turns out that the vanishing points of these ζ_i 's are precisely the threshold points of capacities. Each specific heat involves one ζ_i . Explicitly,

$$\begin{aligned} C_{Q,J} &= T \frac{\partial S}{\partial T} \Big|_{Q,J} = \frac{8\pi^2 r_0 (a^2 + r_0^2)^2 (q^2 + r_0^2 + \lambda^2 r_0^4 + a^2(1 + \lambda^2 r_0^2)) T}{(1 - a^2\lambda^2) \zeta_2} \\ C_{Q,\Omega} &= T \frac{\partial S}{\partial T} \Big|_{Q,\Omega} = \frac{8\pi^2 r_0^3 (a^2 + r_0^2)^2 (1 + \lambda^2 r_0^2) T}{(1 - a^2\lambda^2) \zeta_4}, \\ C_{\Phi,J} &= T \frac{\partial S}{\partial T} \Big|_{\Phi,J} = \frac{8\pi^2 r_0 (r_0^2 + a^2)^2 T}{(1 - a^2\lambda^2) \zeta_3} \left(a^4(1 + \lambda^2 r_0^2) \right. \\ &\quad \left. + r_0^2(q^2 + r_0^2 + \lambda^2 r_0^4) + a^2(q^2(3 + 2\lambda^2 r_0^2) + 2(r_0^2 + \lambda^2 r_0^4)) \right), \\ C_{\Phi,\Omega} &= T \frac{\partial S}{\partial T} \Big|_{\Phi,\Omega} = \frac{8\pi^2 r_0^3 (a^2 + r_0^2)^2 (1 + \lambda^2 r_0^2) T}{(1 - a^2\lambda^2) \zeta_1} \end{aligned} \quad (\text{B.3})$$

Only two charge capacitances and two moments of inertia involve ζ_i ,

$$\tilde{C}_{T,J} = \frac{\partial Q}{\partial \Phi} \Big|_{T,J} = \frac{(a^2 + r_0^2)^2 \zeta_2}{(1 - a^2\lambda^2) r_0 \zeta_3},$$

$$\begin{aligned}
\tilde{C}_{T,\Omega} &= \frac{\partial Q}{\partial \Phi} \Big|_{T,\Omega} = \frac{(a^2 + r_0^2) \zeta_4}{(1 - a^2 \lambda^2) r_0 \zeta_1}, \\
\hat{C}_{T,Q} &= \frac{\partial J}{\partial \Omega} \Big|_{T,Q} = -\frac{(a^2 + r_0^2)^2 \zeta_2}{2(1 - a^2 \lambda^2)^3 r_0 \zeta_4}, \\
\hat{C}_{T,\Phi} &= \frac{\partial J}{\partial \Omega} \Big|_{T,\Phi} = -\frac{(a^2 + r_0^2) \zeta_3}{2(1 - a^2 \lambda^2)^3 r_0 \zeta_1}.
\end{aligned} \tag{B.4}$$

Note that ζ_2 and ζ_3 can appear in numerators as well. The charge capacitances $\tilde{C}_{S,J}$, $\tilde{C}_{S,\Omega}$ and moments of inertia $\hat{C}_{S,Q}$, $\hat{C}_{S,\Phi}$ do not involve ζ_i . They have neither zero nor pole and we shall not present them.

C Capacities of Kerr-AdS₅ Black holes

In this appendix, we present some detailed results discussed in section 5. The thermodynamical curvatures are in general too complex to present. Here we only give the poles:

$$\begin{aligned}
R^{(M)} = -R^{(\bar{M})} &\sim \frac{1}{\zeta_1^2}, & R^{(F)} = -R^{(\bar{F})} &\sim \frac{1}{\zeta_2^2 \zeta_3^2}, \\
R^{(L_a)} = -R^{(\bar{L}_a)} &\sim \frac{1}{\zeta_4^2 \zeta_6^2}, & R^{(L_b)} = -R^{(\bar{L}_b)} &\sim \frac{1}{\zeta_5^2 \zeta_7^2},
\end{aligned} \tag{C.1}$$

where

$$\begin{aligned}
\zeta_1 &= a^2 b^2 - (1 + a^2 \lambda^2 + b^2 \lambda^2) r_0^4 + 2 \lambda^2 r_0^6, \\
\zeta_2 &= -a^2 b^2 - (a^2 + b^2) r_0^2 + 3 r_0^4, \\
\zeta_3 &= 5 a^4 b^4 + 9 a^4 b^2 r_0^2 + 9 a^2 b^4 r_0^2 + 20 a^2 b^2 r_0^4 + 3 a^2 r_0^6 + 3 b^2 r_0^6 - r_0^8 \\
&\quad + a^2 b^2 \lambda^6 r_0^4 (a^2 + r_0^2) (b^2 + r_0^2) (3(a^2 + b^2) + 10 r_0^2) \\
&\quad + \lambda^2 (3 a^4 b^4 (a^2 + b^2) + 3 a^2 b^2 (a^4 + 10 a^2 b^2 + b^4) r_0^2 \\
&\quad + 39 a^2 b^2 (a^2 + b^2) r_0^4 + 2 (2 a^4 + 33 a^2 b^2 + 2 b^4) r_0^6 \\
&\quad + 14 (a^2 + b^2) r_0^8 + 2 r_0^{10}) + \lambda^4 (a^6 b^6 + 9 a^4 b^4 (a^2 + b^2) r_0^2 \\
&\quad + a^2 b^2 (9 a^4 + 46 a^2 b^2 + 9 b^4) r_0^4 + (a^2 + b^2) (a^4 + 43 a^2 b^2 + b^4) r_0^6 \\
&\quad + (3 a^4 + 49 a^2 b^2 + 3 b^4) r_0^8 + 6 (a^2 + b^2) r_0^{10}), \\
\zeta_4 &= -a^2 (1 - b^2 \lambda^2) + (3 + b^2 \lambda^2) r_0^2, \\
\zeta_5 &= -b^2 (1 - a^2 \lambda^2) + (3 + a^2 \lambda^2) r_0^2, \\
\zeta_6 &= -b^2 (a^2 + r_0^2) (3 a^2 + 3 a^2 \lambda^4 r_0^4 + \lambda^2 (a^4 + 8 a^2 r_0^2 + r_0^4)) \\
&\quad + r_0^4 (3 a^2 - r_0^2 + 2 \lambda^2 (2 a^4 + 7 a^2 r_0^2 + r_0^4) + a^2 \lambda^4 (a^4 + 3 a^2 r_0^2 + 6 r_0^4)), \\
\zeta_7 &= -a^2 (b^2 + r_0^2) (3 b^2 + 3 b^2 \lambda^4 r_0^4 + \lambda^2 (b^4 + 8 b^2 r_0^2 + r_0^4)) \\
&\quad + r_0^4 (3 b^2 - r_0^2 + 2 \lambda^2 (2 b^4 + 7 b^2 r_0^2 + r_0^4) + b^2 \lambda^4 (b^4 + 3 b^2 r_0^2 + 6 r_0^4)).
\end{aligned} \tag{C.2}$$

Note that ζ_1 , ζ_2 and ζ_3 are symmetric under the interchange of a and b , whilst ζ_4 and ζ_5 , ζ_6 and ζ_7 interchange with each other.

There are four specific heat, given by

$$\begin{aligned}
C_{J_a, J_b} &= \frac{\pi^3 (3 + b^2 \lambda^2 + a^2 \lambda^2 + (1 - b^2 \lambda^2)) (a^2 + r_0^2)^3 (b^2 + r_0^2)^3 T}{(1 - a^2 \lambda^2) (1 - b^2 \lambda^2) \zeta_3}, \\
C_{J_a, \Omega_b} &= \frac{T \zeta_5 \pi^3 (a^2 + r_0^2)^3 (b^2 + r_0^2)}{\zeta_6 (1 - a^2 \lambda^2) (1 - b^2 \lambda^2)}, \\
C_{\Omega_a, J_b} &= \frac{T \zeta_4 \pi^3 (a^2 + r_0^2) (b^2 + r_0^2)^3}{\zeta_7 (1 - a^2 \lambda^2) (1 - b^2 \lambda^2)}, \\
C_{\Omega_a, \Omega_b} &= \frac{T \zeta_2 \pi^3 (a^2 + r_0^2) (b^2 + r_0^2)}{\zeta_1 (1 - a^2 \lambda^2) (1 - b^2 \lambda^2)}. \tag{C.3}
\end{aligned}$$

There are four moments of inertia associated with index a . They are given by

$$\begin{aligned}
\hat{C}_{T, J_b} &= \frac{\zeta_3 \pi (a^2 + r_0^2) (b^2 + r_0^2)}{4 \zeta_7 (1 - a^2 \lambda^2)^3 (1 - b^2 \lambda^2) r_0^2}, \\
\hat{C}_{T, \Omega_b} &= \frac{\zeta_6 \pi (a^2 + r_0^2) (b^2 + r_0^2)}{4 \zeta_1 (1 - a^2 \lambda^2)^3 (1 - b^2 \lambda^2) r_0^2}, \\
\hat{C}_{S, J_b} &= \frac{\pi (3 + b^2 \lambda^2 + a^2 \lambda^2 (1 - b^2 \lambda^2)) (a^2 + r_0^2)^3 (b^2 + r_0^2)}{4 \zeta_4 (1 - a^2 \lambda^2)^3 (1 - b^2 \lambda^2) r_0^2}, \\
\hat{C}_{S, \Omega_b} &= \frac{\zeta_5 \pi (a^2 + r_0^2)^3 (b^2 + r_0^2)}{4 \zeta_2 (1 - a^2 \lambda^2)^3 (1 - b^2 \lambda^2) r_0^2}. \tag{C.4}
\end{aligned}$$

The moments of inertia associated with index b are given above, but with (a, b) , (ζ_4, ζ_5) and (ζ_6, ζ_7) interchanged.

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